EQUATION OF STATE OF MATTER AT SUPERNUCLEAR DENSITY

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We shall consider briefly the possible nature of the equation of state of matter at supernuclear density. This is of interest in such subjects as cosmology and gravitational collapse. By supernuclear density we mean a density exceeding that of ordinary nuclear matter. Since nucleons in nuclei are spaced approximately 1 fermi apart, we have in mind proper densities of $\rho > 10^{15}$ g cm^{-3}. Zel'dovitch (1962) has shown on the basis of a vector meson model that as the supernuclear density increases in an isotropic fluid the pressure tends to equal the energy density. Our aim is to show that it is unlikely that Zel'dovitch's conclusion is true.

The equation of state can be expressed in the form

$$p = (\nu - 1)\,\epsilon \tag{1}$$

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where ρ is the pressure and $\epsilon = \rho c^2$ is the energy density. Both ρ and ν are matrices which can be diagonalized for microscopic elements of fluid. For a photon or neutrino gas, in which the particles have zero rest mass,

$$\nu = \frac{q + 1}{q} \tag{2}$$

where $q = 1$, 2, or 3 is the number of spatial degrees of freedom. When the gas is isotropic, $q = 3$, and therefore according to (2), $\nu = 4/3$, and $\rho = \epsilon/3$. For a gas containing a class of particles of energy $E_i = \gamma_i m_i c^2$, where m_i is their rest mass and γ_i is the ratio of energy to rest energy,

$$\nu_i = \frac{\gamma_i^2(q + 1) - 1}{q\,\gamma_i^2} \tag{3}$$

if the interaction energy is ignored. By regarding the energy contribution from the various interaction fields as a gas of real and virtual bosons, the effective value of ν for the fluid can be evaluated by integrating (3) over the various distribution functions of the particles. Since $\nu_i \le (q + 1)/q$ for all classes of particles, we have in general

$$1 \le \nu \le \frac{q+1}{q} \tag{4}$$

for all fluids. The lower limit is for a gas of zero pressure, and the upper limit is for a gas containing only particles of zero rest mass. It is also noticed that the velocity of sound is

$$v_s = c\,(dp/d\varepsilon)^{\frac{1}{2}} = c\,(\nu - 1)^{\frac{1}{2}} \le c\,q^{-\frac{1}{2}} \tag{5}$$

in the appropriate direction, and has its maximum value of $v_s = c$ at $q = 1$, as one would expect. In an isotropic gas, which is of main interest, (4) becomes $1 \le \nu \le 4/3$ and therefore $p < \varepsilon/3$, and gives $v_s < c/3^{\frac{1}{2}}$.

In spite of the complexity of the particle states at supernuclear density our classical picture indicates that $1 \leq \nu \leq 4/3$ must still hold true in an istropic fluid. It would also seem reasonable to suppose that at supernuclear density ν has a value close to $4/3$. Using the adiabatic relativistic equation:

$$d\,(\varepsilon V) + \rho dV = 0 \tag{6}$$

for a variable volume V, we have from (1) that

$$\left(\frac{d}{dV} + \frac{\nu - 1}{V}\right)\varepsilon V = 0 \tag{7}$$

Assuming ν is constant in a given density range, it follows that

$$\varepsilon \propto V^{-\nu} \tag{8}$$

and if ν has a value close to $4/3$, then $\epsilon \propto V^{-4/3}$, as in a relativistic ideal gas.

The relatively simple picture outlined above has been thrown into a state of confusion by Zel'dovitch's (1962) suggestion that strong interactions increase the value of ν, and that in the limit of high density in an isotropic fluid, $\nu \rightarrow 2$, and therefore $\rho \rightarrow \epsilon$ and $v_s \rightarrow c$. It is argued that the range of variation of ν is $1 \leq \nu \leq 2$, and not $1 \leq \nu \leq 4/3$ as given by the previous argument. Anisotropy in such a fluid is now impossible as this would imply that the velocity of sound could exceed the velocity of light. Zel'dovitch's suggestion is based on a simple model in which stationary baryons interact through a vector meson field.

The Proca equations (Morse and Feshbach, 1953) for a vector meson (unit-spin) field are

$$\nabla^2 \underset{\sim}{A} - \frac{1}{c^2}\frac{\partial^2 \underset{\sim}{A}}{\partial t^2} - \lambda^{-2}\underset{\sim}{A} + \frac{4\pi}{c}\underset{\sim}{J} = 0 \tag{9a}$$

$$\nabla^2 \phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} - \lambda^{-2}\phi + 4\pi\rho_e = 0 \tag{9b}$$

with the gauge condition

$$\nabla . \underset{\sim}{A} + \frac{1}{c} \frac{\partial \Phi}{\partial t} = 0 \tag{9c}$$

where $\underset{\sim}{A}$ and Φ are the potentials, and $\underset{\sim}{j}$ and ρ_e are the current and charge densities. These Lorentz invariant equations are for "hyperphotons" (vector mesons) of rest mass $\mu = \hbar/\lambda c$. When $\mu = 0$, they reduce to Maxwell's equations for photons of zero rest mass. Zel'devitch assumes static conditions and the Proca equations therefore become

$$\nabla^2 \Phi - \lambda^{-2} \Phi = 0 \tag{10}$$

Hence, if $g^2/\hbar c$ is the coupling constant, where g is the baryon "charge", the solution is the Yukawa-type potential

$$\Phi = g r^{-1} e^{-r/\lambda} \tag{11}$$

Zel'dovitch then shows that if there are n baryons per unit volume, and m is the baryon mass, the energy per baryon is

$$E = mc^2 + \tfrac{1}{2} g^2 n \int r^{-1} e^{-r/\lambda} dV = mc^2 + 2\pi g^2 n \lambda^2$$

and the energy density is therefore

$$\epsilon = nmc^2 + 2\pi g^2 n^2 \lambda^2 \tag{12}$$

If it is assumed that the range λ of the interaction is independent of density, it follows from (6) that the pressure is

$$p = - \frac{d}{dV} (\epsilon V) = 2\pi g^2 n^2 \lambda^2 \tag{13}$$

Comparing (12) and (13) it is seen that in the limit of high density we have $\rho \to \epsilon$, and according to (1) ν therefore approaches a value of 2. The assumption that the potential (11) is density independent means that the range λ of the interaction at high densities is large compared with the interbaryon distance. The interaction therefore tends to become long-range and collective rather than short-range, thus accounting for ν approaching the value of 2.

The following comments must be made regarding Zel'dovitch's treatment. At supernuclear densities it is physically unrealistic to ignore the high energy of the baryons imposed by the exclusion principle. In ordinary nuclear matter the ideal gas laws provide a good starting point for calculating the energy shift (Weisskopf 1950, Bell and Squires 1961), and potential functions are employed to obtain improved approximations. When the internucleon distance is less than the nucleon Compton wavelength, or $\rho > \sim 10^{17}$ gcm^{-3}, the idea of stationary baryons is completely unacceptable since the nucleons now have energies exceeding 1 GeV and there is copious production, among other things, of nucleons and hyperons (including their antiparticles). A further unsatisfactory feature of the model is that it singles out unit-spin mesons and neglects the rest of the meson multiplets.

Let us, however, ignore the unsatisfactory nature of Zel'dovitch's model and accept the potential (11) as a crude phenomenological description of short-range interactions. As the density increases (and the available energy involved in the interactions also increases) the higher masses of the meson hierarchy will progressively predominate in the interactions. In effect, λ will get smaller and the short-range character of strong interactions will be preserved. Thus, the potential (11) can no longer be regarded as density independent. As an example, Levy (1952) points out that the higher order potentials behave as exp $(-xr/\lambda)$ and involve the exchange of x mesons. It is to be expected that the range of the predominant interaction is comparable with the interbaryon distance, and therefore λ is related to density by an expression of the kind $\lambda = \alpha n^{-\frac{1}{3}}$, where α is of the order unity. Equations (12) and (13) now become

$$\epsilon = nmc^2 + 2\pi g^2\alpha^2 n^{\frac{4}{3}} \tag{14}$$

$$\rho = \frac{2\pi}{3}\, g^2\alpha^2 n^{\frac{4}{3}} \tag{15}$$

and these equations have the merit that they yield the physically acceptable result that ν has a maximum value of $4/3$.

Despite the complexity of the interactions in matter at supernuclear density it is unlikely that our earlier arguments concerning the maximum possible value of ν are in error. Although Zel'dovitch's model possesses several unacceptable features, it is nevertheless interesting to notice that if it is applied in a slightly more realistic manner then the maximum possible value of ν in an isotropic fluid is also $4/3$.

REFERENCES

Bell, J. S., and Squires, E. J., 1961, Advances in Physics, **10**, 211

Levy, M. M., 1952, Physical Review, **88**, 725

Morse, P. M., and Feshbach, H., 1953, Methods of Theoretical Physics, (McGraw-Hill, New York), p. 221, 271.

Weisskopf, V. F., 1950, Helv. Phys. Acta, **23**, 187

Zel'dovitch, Ya., B., 1962, Soviet Physics JETP, **14**, 1143